

APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL VIA A ČEBYŠEV TYPE FUNCTIONAL

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ABSTRACT. Some new sharp upper bounds for the absolute value of the error functional $D(f, u)$ in approximating the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the quantity $[u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt$ are given.

1. INTRODUCTION

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the simpler quantity

$$[u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

provided that both integrals exist, Dragomir and Fedotov introduced in [8] the following *error functional of Čebyšev type*

$$(1.1) \quad D(f; u) = \int_a^b f(t) du(t) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

and pointed out the following sharp upper bound for $|D(f; u)|$, namely

$$(1.2) \quad |D(f; u)| \leq \frac{1}{2} L (M - m) (b - a),$$

provided the *integrator* $u : [a, b] \rightarrow \mathbb{R}$ is L -Lipschitzian on $[a, b]$, i.e., $|u(x) - u(y)| \leq L|x - y|$ for any $x, y \in [a, b]$ and the *integrand* $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ and satisfies the boundedness condition

$$(1.3) \quad -\infty < m \leq f(x) \leq M < \infty \quad \text{for a.e. } x \in [a, b].$$

The multiplicative constant $\frac{1}{2}$ in (1.2) is best possible in the sense that it cannot be replaced by a smaller constant.

In the follow-up paper [9], the authors provided a different bound, namely

$$(1.4) \quad |D(f; u)| \leq \frac{1}{2} K (b - a) \bigvee_a^b(u),$$

provided that f is K -Lipschitzian and u is of bounded variation on $[a, b]$.

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The result (1.4) was improved in [4] for the case of monotonic nondecreasing functions. We have shown in this case that

$$(1.5) \quad |D(f; u)| \leq \frac{1}{2} K(b-a) [u(b) - u(a) - K(u)] \\ \left(\leq \frac{1}{2} K(b-a) [u(b) - u(a)] \right),$$

where

$$K(u) := \frac{4}{(b-a)^2} \int_a^b u(t) \left(t - \frac{a+b}{2} \right) dt \geq 0.$$

In (1.5) the constant $\frac{1}{2}$ is best possible in both inequalities.

For other sharp bounds on the error functional $D(f; u)$, see the recent papers [6], [7], and [10]. For other inequalities for the Riemann-Stieltjes integral, see [1] and [2].

The main aim of this paper is to further investigate the error functional $D(f; u)$. Two representations are given. These are applied to obtain some inequalities for $D(f; u)$ which improve earlier results.

Applications for the classical *Čebyšev functional* $C(f, g)$, where

$$(1.6) \quad C(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt,$$

and f, g are integrable and belonging to different classes of functions, are also provided.

2. REPRESENTATION RESULTS

For a function $g : [a, b] \rightarrow \mathbb{R}$, consider the *generalised trapezoid error transform* $\Phi_g : [a, b] \rightarrow \mathbb{R}$ given by

$$(2.1) \quad \Phi_g(t) := \frac{1}{b-a} [(b-t)g(a) + (t-a)g(b)] - g(t), \quad t \in [a, b]$$

and if g is Lebesgue integrable, the *Ostrowski transform*, which is the error of approximating the function by its integral mean, defined by:

$$(2.2) \quad \Theta_g(t) := g(t) - \frac{1}{b-a} \int_a^b g(s) ds, \quad t \in [a, b].$$

We also define the kernel $Q : [a, b]^2 \rightarrow \mathbb{R}$,

$$(2.3) \quad Q(t, s) := \begin{cases} t-b & \text{if } a \leq s \leq t \leq b, \\ t-a & \text{if } a \leq t < s \leq b. \end{cases}$$

The following representation result in terms of Θ_g and Q may be stated:

Lemma 1. *If $f, u : [a, b] \rightarrow \mathbb{R}$ are bounded functions and such that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ and the Riemann integral $\int_a^b f(t) dt$ exist, then we have the representation:*

$$(2.4) \quad D(f; u) = \int_a^b \Theta_f(s) du(s) = \frac{1}{b-a} \int_a^b \left(\int_a^b Q(t, s) df(t) \right) du(s).$$

Proof. We have by the definition of Q and integrating by parts in the Riemann-Stieltjes integral that

$$\begin{aligned}
& \frac{1}{b-a} \int_a^b \left(\int_a^b Q(t, s) df(t) \right) du(s) \\
&= \frac{1}{b-a} \int_a^b \left[\int_a^s (t-a) df(t) + \int_s^b (t-b) df(t) \right] du(s) \\
&= \frac{1}{b-a} \int_a^b \left[f(t)(t-a) \Big|_a^s - \int_a^s f(t) dt + (t-b)f(t) \Big|_s^b - \int_s^b f(t) dt \right] du(s) \\
&= \frac{1}{b-a} \int_a^b \left[f(s)(s-a) - \int_a^s f(t) dt + (b-s)f(s) - \int_s^b f(t) dt \right] du(s) \\
&= \int_a^b \Theta_f(s) du(s)
\end{aligned}$$

and the second inequality is proved.

The first identity is obvious by the definition of $D(f; u)$. \square

The following corollary can be stated about the representation of the Čebyšev functional $C(f, g)$ defined in (1.6).

Corollary 1. Assume that $f, g : [a, b] \rightarrow \mathbb{R}$ are Riemann integrable on $[a, b]$, then

$$\begin{aligned}
(2.5) \quad C(f, g) &= \frac{1}{b-a} \int_a^b \Theta_f(s) g(s) ds \\
&= \frac{1}{(b-a)^2} \int_a^b \left(\int_a^b Q(t, s) df(t) \right) g(s) ds.
\end{aligned}$$

Proof. It is well known (see for instance [3, Theorem 7.33, p. 162] that if g is Riemann integrable and $u(t) = \int_a^t g(s) ds$, then for any f a Riemann integrable function we have that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists and $\int_a^b f(t) du(t) = \int_a^b f(t) g(t) dt$. Therefore, we have $D(f; u) = (b-a) C(f, g)$ and

$$\int_a^b \left(\int_a^b Q(t, s) df(t) \right) du(s) = \int_a^b \left(\int_a^b Q(t, s) df(t) \right) g(s) ds.$$

\square

The second representation of $D(f; u)$ is incorporated in

Lemma 2. With the assumptions in Lemma 1, we have

$$(2.6) \quad D(f; u) = \int_a^b \Phi_u(t) df(t) = \frac{1}{b-a} \int_a^b \left(\int_a^b Q(t, s) du(s) \right) df(t),$$

where Q is defined by (2.3).

Proof. By the Fubini type theorem for the Riemann-Stieltjes integral (see for instance [3, Theorem 7.41, p. 167]) we have that

$$\int_a^b \left(\int_a^b Q(t, s) du(s) \right) df(t) = \int_a^b \left(\int_a^b Q(t, s) df(t) \right) du(s),$$

and the equality between the first and the last term in (2.6) is proved.

Now, observe that

$$\begin{aligned} \int_a^b Q(t, s) du(s) &= \int_a^t (t-b) du(s) + \int_t^b (t-a) du(s) \\ &= (t-b)[u(t) - u(a)] + (t-a)[u(b) - u(t)] \\ &= (b-a)\Phi_u(t), \end{aligned}$$

for any $t \in [a, b]$, and then integrating over $f(t)$, we deduce the second inequality in (2.6). \square

Corollary 2. *Assume that f and g are Riemann integrable on $[a, b]$, then we have*

$$(2.7) \quad C(f, g) = \frac{1}{b-a} \int_a^b \tilde{\Phi}_g(t) df(t) = \frac{1}{(b-a)^2} \int_a^b \left(\int_a^b Q(t, s) g(s) ds \right) df(t),$$

where

$$(2.8) \quad \tilde{\Phi}_g(t) = \Phi_{\int_a^\cdot g}(t) = \frac{t-a}{b-a} \int_a^b g(s) ds - \int_a^t g(s) ds, \quad t \in [a, b].$$

3. BOUNDS IN THE CASE WHEN u IS OF BOUNDED VARIATION

The following lemma is of interest in itself.

Lemma 3. *If $p : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and $v : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$, then*

$$\begin{aligned} (3.1) \quad \left| \int_a^b p(t) dv(t) \right| &\leq \int_a^b |p(t)| d \bigvee_a^t(v) \\ &\leq \left[\bigvee_a^b(v) \right]^{\frac{1}{q}} \left\{ \int_a^b |p(t)|^p d \left[\bigvee_a^t(v) \right] \right\}^{\frac{1}{p}} \\ &\leq \max_{t \in [a, b]} |p(t)| \bigvee_a^b(v), \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since the Stieltjes integral $\int_a^b p(t) dv(t)$ exists, then for any division $I_n : a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ with the norm $v(I_n) := \max_{i \in \{0, \dots, n-1\}} (t_{i+1} - t_i) \rightarrow 0$ and for any intermediate points $\xi_i \in [t_i, t_{i+1}]$, $i \in \{0, \dots, n-1\}$ we have

$$\begin{aligned} (3.2) \quad \left| \int_a^b p(t) dv(t) \right| &= \left| \lim_{v(I_n) \rightarrow 0} \sum_{i=0}^{n-1} p(\xi_i) [v(t_{i+1}) - v(t_i)] \right| \\ &\leq \lim_{v(I_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i)| |v(t_{i+1}) - v(t_i)|. \end{aligned}$$

However,

$$(3.3) \quad |v(t_{i+1}) - v(t_i)| \leq \bigvee_{t_i}^{t_{i+1}}(v) = \bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v),$$

for any $i \in \{0, \dots, n-1\}$ and by (3.3) we have

$$\begin{aligned} \left| \int_a^b p(t) dv(t) \right| &\leq \lim_{v(I_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i)| \left[\bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v) \right] \\ &= \int_a^b |p(t)| d \left[\bigvee_a^t(v) \right], \end{aligned}$$

and the last Riemann-Stieltjes integral exists since $|p|$ is continuous and $\bigvee_a^t(v)$ is monotonic nondecreasing.

The last part follows from the following Hölder type inequality

$$(3.4) \quad \left| \int_a^b g(t) dv(t) \right| \leq [v(b) - v(a)]^{\frac{1}{q}} \left[\int_a^b |g(t)|^p dv(t) \right]^{\frac{1}{p}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that holds for any continuous function $g : [a, b] \rightarrow \mathbb{R}$ and any monotonic nondecreasing function $v : [a, b] \rightarrow \mathbb{R}$. The details are omitted. \square

The following result holds.

Theorem 1. Assume that $f, u : [a, b] \rightarrow \mathbb{R}$ are of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists. Then

$$\begin{aligned} (3.5) \quad |D(f; u)| &\leq \frac{1}{b-a} \left[\int_a^b \bigvee_a^s(f) (2s-a-b) d \left(\bigvee_a^s(u) \right) \right. \\ &\quad \left. + 2 \int_a^b \left(\bigvee_a^s(u) \cdot \bigvee_a^s(f) \right) ds - \bigvee_a^b(u) \int_a^b \left(\bigvee_a^s(f) \right) ds \right] \\ &\leq \frac{1}{b-a} \int_a^b \bigvee_a^s(f) (2s-a-b) d \left(\bigvee_a^s(u) \right) \\ &\quad + \frac{1}{b-a} \int_a^b \left(\bigvee_a^s(u) \cdot \bigvee_a^s(f) \right) ds \\ &\leq \frac{1}{b-a} \int_a^b \bigvee_a^s(f) (2s-a-b) d \left(\bigvee_a^s(u) \right) + \bigvee_a^b(u) \cdot \bigvee_a^b(f). \end{aligned}$$

Proof. Utilising the identity (2.4) and the first inequality in (3.1) we have:

$$\begin{aligned} (3.6) \quad |D(f; u)| &\leq \frac{1}{b-a} \int_a^b \left| \int_a^b Q(t, s) df(t) \right| d \left(\bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b \left| \int_a^s (t-a) df(t) + \int_s^b (t-b) df(t) \right| d \left(\bigvee_a^s(u) \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{b-a} \int_a^b \left[\left| \int_a^s (t-a) df(t) \right| + \left| \int_s^b (t-b) df(t) \right| \right] d \left(\bigvee_a^s(u) \right) \\ &=: I. \end{aligned}$$

Since f is of bounded variation, then by the same inequality in (3.1) we have

$$\begin{aligned} \left| \int_a^s (t-a) df(t) \right| &\leq \int_a^s (t-a) d \left(\bigvee_a^t(f) \right) \\ &= \bigvee_a^s(f) \cdot (s-a) - \int_a^s \left(\bigvee_a^t(f) \right) dt \end{aligned}$$

and

$$\begin{aligned} \left| \int_s^b (t-b) df(t) \right| &\leq \int_s^b (t-b) d \bigvee_s^t(f) = \int_s^b \left(\bigvee_s^t(f) \right) dt \\ &= \int_s^b \left[\bigvee_a^t(f) - \bigvee_a^s(f) \right] dt \\ &= \int_s^b \left(\bigvee_a^t(f) \right) dt - (b-s) \bigvee_a^s(f) \end{aligned}$$

which gives that

(3.7)

$$\begin{aligned} I &\leq \frac{1}{b-a} \int_a^b \left[\bigvee_a^s(f) (s-a) - \int_a^s \left(\bigvee_a^t(f) \right) dt \right. \\ &\quad \left. + \int_s^b \left(\bigvee_a^t(f) \right) dt - (b-s) \bigvee_a^s(f) \right] d \left(\bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b \left[\bigvee_a^s(f) (2s-a-b) - \int_a^s \left(\bigvee_a^t(f) \right) dt \right. \\ &\quad \left. + \int_s^b \left(\bigvee_a^t(f) \right) dt \right] d \left(\bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b (2s-a-b) \bigvee_a^s(f) d \left(\bigvee_a^s(u) \right) \\ &\quad + \frac{1}{b-a} \int_a^b \left[\int_a^b \left(\bigvee_a^t(f) \right) dt - 2 \int_a^s \left(\bigvee_a^t(f) \right) dt \right] d \left(\bigvee_a^s(u) \right) \\ &= \frac{1}{b-a} \int_a^b (2s-a-b) \bigvee_a^s(f) d \left(\bigvee_a^s(u) \right) + \frac{1}{b-a} \int_a^b \left(\bigvee_a^t(f) \right) dt \cdot \bigvee_a^b(u) \\ &\quad - \frac{2}{b-a} \int_a^b \left(\int_a^s \left(\bigvee_a^t(f) \right) dt \right) d \left(\bigvee_a^s(u) \right). \end{aligned}$$

However, integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} & \int_a^b \left(\int_a^s \left(\bigvee_a^t(f) \right) dt \right) d \left(\bigvee_a^s(u) \right) \\ &= \int_a^s \left(\bigvee_a^t(f) \right) dt \cdot \bigvee_a^s(u) \Big|_a^b - \int_a^b \bigvee_a^s(u) \cdot \bigvee_a^s(f) ds \\ &= \bigvee_a^b(u) \cdot \int_a^b \left(\bigvee_a^t(f) \right) dt - \int_a^b \bigvee_a^s(u) \cdot \bigvee_a^s(f) ds \end{aligned}$$

Inserting this value in the expression of I from (3.7) we deduce the first inequality in (3.5).

The other inequalities are obvious. \square

The following result may be stated as well.

Theorem 2. *If $u : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and $f : [a, b] \rightarrow \mathbb{R}$ is L -Lipschitzian, then*

$$\begin{aligned} (3.8) \quad |D(f; u)| &\leq L \left[\frac{1}{2} (b-a) \bigvee_a^b(u) - \frac{2}{b-a} \int_a^b \left(\bigvee_a^s(u) \right) \left(s - \frac{a+b}{2} \right) ds \right] \\ &\leq \frac{1}{2} L (b-a) \bigvee_a^b(u). \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. It is well known that if $p : [\alpha, \beta] \rightarrow \mathbb{R}$ is L -Lipschitzian and $v : [\alpha, \beta] \rightarrow \mathbb{R}$ is Riemann integrable, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(s) dv(s)$ exists and $\left| \int_\alpha^\beta p(s) dv(s) \right| \leq L \int_\alpha^\beta |p(s)| ds$. Utilising this property, we then have

$$\begin{aligned} \left| \int_a^s (t-a) df(t) \right| &\leq L \int_a^s (t-a) dt = \frac{L}{2} (s-a)^2, \\ \left| \int_s^b (t-b) df(t) \right| &\leq L \int_s^b (b-t) dt = \frac{L}{2} (b-s)^2. \end{aligned}$$

Therefore, by the relation (3.6) we have

$$\begin{aligned} I &\leq \frac{L}{2(b-a)} \int_a^b \left[(b-s)^2 + (s-a)^2 \right] d \left(\bigvee_a^s(u) \right) \\ &= \frac{L}{2(b-a)} \left[\left[(b-s)^2 + (s-a)^2 \right] \bigvee_a^s(u) \Big|_a^b - 2 \int_a^b \bigvee_a^s(u) (2s-a-b) ds \right] \\ &= \frac{L}{2(b-a)} \left[(b-a)^2 \bigvee_a^b(u) - 4 \int_a^b \bigvee_a^s(u) \left(s - \frac{a+b}{2} \right) ds \right] \end{aligned}$$

and the first inequality in (3.8) is proved.

To prove the last part, we use the Čebyšev inequality which states that for two nondecreasing functions g and h ,

$$\frac{1}{b-a} \int_a^b g(s) h(s) ds \geq \frac{1}{b-a} \int_a^b g(s) ds \cdot \frac{1}{b-a} \int_a^b h(s) ds.$$

Then

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \bigvee_a^s(u) \left(s - \frac{a+b}{2}\right) ds \\ & \geq \frac{1}{b-a} \int_a^b \left(\bigvee_a^s(u)\right) ds \cdot \frac{1}{b-a} \int_a^b \left(s - \frac{a+b}{2}\right) ds \end{aligned}$$

and since $\int_a^b \left(s - \frac{a+b}{2}\right) ds = 0$, the inequality is proved.

For the sharpness of the constant, we consider the functions $f(t) = t - \frac{a+b}{2}$, $t \in [a, b]$ and $u : [a, b] \rightarrow \mathbb{R}$ defined by

$$u(t) := \begin{cases} 1 & \text{if } t = a \\ 0 & \text{if } t \in (a, b) \\ 1 & \text{if } t = b. \end{cases}$$

Then f is Lipschitzian with $L = 1$ and u is of bounded variation on $[a, b]$. We have $\bigvee_a^s(u) = 1$, $s \in (a, b)$ and $\bigvee_a^b(u) = 2$. Also,

$$\begin{aligned} D(f; u) &= \int_a^b f(t) du(t) - \frac{u(b) - u(a)}{b-a} \int_a^b f(t) dt \\ &= \int_a^b f(t) du(t) \\ &= f(t) u(t) \Big|_a^b - \int_a^b u(t) df(t) = b - a \end{aligned}$$

and

$$\int_a^b \left(\bigvee_a^s(u)\right) \left(s - \frac{a+b}{2}\right) ds = \int_a^b \left(s - \frac{a+b}{2}\right) ds = 0.$$

Replacing the values in (3.8) we get in all sides the same quantity $b - a$. This shows that the constant $\frac{1}{2}$ is best possible in both inequalities. \square

Remark 1. *The inequality between the first and last term in (3.8) was firstly discovered by Dragomir and Fedotov in [9] where they also showed the sharpness of the constant $\frac{1}{2}$.*

The following result may be stated as well.

Theorem 3. *Assume that $u : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and $f : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing and such that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$*

exists. Then,

$$(3.9) \quad |D(f; u)| \leq \frac{1}{b-a} \left[\int_a^b (2s-a-b) f(s) d\left(\bigvee_a^s(u)\right) + 2 \int_a^b \left(\bigvee_a^s(u)\right) f(s) ds - \int_a^b f(s) ds \cdot \bigvee_a^b(u) \right].$$

Proof. It is well known that if the Stieltjes integrals $\int_\alpha^\beta p(t) dv(t)$ and $\int_\alpha^\beta |p(t)| dv(t)$ exist and v is monotonic nondecreasing on $[\alpha, \beta]$, then

$$\left| \int_\alpha^\beta p(t) dv(t) \right| \leq \int_\alpha^\beta |p(t)| dv(t).$$

Utilising this property we then have

$$\left| \int_a^s (t-a) df(t) \right| \leq \int_a^s (t-a) df(t) = (s-a)f(s) - \int_a^s f(t) dt$$

and

$$\left| \int_s^b (t-b) df(t) \right| \leq \int_s^b (t-b) df(t) = \int_s^b f(t) dt - (b-s)f(s)$$

for any $s \in [a, b]$.

Utilising the relation (3.6), we obtain

(3.10)

$$\begin{aligned} I &\leq \frac{1}{b-a} \left[\int_a^b \left\{ (s-a)f(s) - \int_a^s f(t) dt + \int_s^b f(t) dt - (b-s)f(s) \right\} d\left(\bigvee_a^s(u)\right) \right] \\ &= \frac{1}{b-a} \left[\int_a^b (2s-a-b) f(s) d\left(\bigvee_a^s(u)\right) + \int_a^b \left(\int_s^b f(t) dt \right) d\left(\bigvee_a^s(u)\right) - \int_a^b \left(\int_a^s f(t) dt \right) d\left(\bigvee_a^s(u)\right) \right] \\ &=: J. \end{aligned}$$

However, integrating by parts in the Riemann-Stieltjes integral, we have:

$$\begin{aligned} \int_a^b \left(\int_s^b f(t) dt \right) d\left(\bigvee_a^s(u)\right) &= \left(\int_s^b f(t) dt \right) \cdot \bigvee_a^s(u) \Big|_a^b - \int_a^b \left(\bigvee_a^s(u) \right) d\left(\int_s^b f(t) dt \right) \\ &= \int_a^b \left(\bigvee_a^s(u) \right) f(s) ds \end{aligned}$$

and

$$\begin{aligned} \int_a^b \left(\int_a^s f(t) dt \right) d \left(\bigvee_a^s(u) \right) &= \left(\int_a^s f(t) dt \right) \cdot \bigvee_a^s(u) \Big|_a^b - \int_a^b \left(\bigvee_a^s(u) \right) d \left(\int_a^s f(t) dt \right) \\ &= \int_a^b f(t) dt \cdot \bigvee_a^b(u) - \int_a^b \left(\bigvee_a^s(u) \right) f(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} J &= \frac{1}{b-a} \left[\int_a^b (2s-a-b) f(s) d \left(\bigvee_a^s(u) \right) \right. \\ &\quad \left. + \int_a^b \left(\bigvee_a^s(u) \right) f(s) ds - \int_a^b f(t) dt \cdot \bigvee_a^b(u) + \int_a^b \left(\bigvee_a^s(u) \right) f(s) ds \right] \\ &= \frac{1}{b-a} \left[\int_a^b (2s-a-b) f(s) d \left(\bigvee_a^s(u) \right) \right. \\ &\quad \left. + 2 \int_a^b \left(\bigvee_a^s(u) \right) f(s) ds - \int_a^b f(s) ds \cdot \bigvee_a^b(u) \right]. \end{aligned}$$

This together with the inequalities (3.6) and (3.10) produces the desired result (3.9). \square

4. BOUNDS IN THE CASE WHEN f IS OF BOUNDED VARIATION

We can state the following result as well.

Theorem 4. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$. If $u : [a, b] \rightarrow \mathbb{R}$ is continuous and such that there exists the constants $L_a, L_b > 0$ and $\alpha, \beta > 0$ with the properties that:

$$(4.1) \quad |u(t) - u(a)| \leq L_a (t-a)^\alpha, \quad |u(t) - u(b)| \leq L_b (b-t)^\beta$$

for any $t \in [a, b]$, then

$$\begin{aligned} (4.2) \quad |D(f; u)| &\leq \frac{1}{b-a} L_a \left[\int_a^b \left(\bigvee_a^t(f) \right) (t-a)^\alpha dt - \alpha \int_a^b \left(\bigvee_a^t(f) \right) (b-t) (t-a)^{\alpha-1} dt \right] \\ &\quad + \frac{1}{b-a} L_b \left[\beta \int_a^b \left(\bigvee_a^t(f) \right) (t-a) (b-t)^{\beta-1} dt - \int_a^b \left(\bigvee_a^t(f) \right) (b-t)^\beta dt \right]. \end{aligned}$$

Proof. Utilising the identity (2.6) and the first inequality in (3.1), we have successively,

(4.3)

$$\begin{aligned}
 |D(f; u)| &\leq \frac{1}{b-a} \int_a^b \left| \int_a^b Q(t, s) du(s) \right| d\left(\bigvee_a^t(f)\right) \\
 &= \frac{1}{b-a} \int_a^b \left| \int_a^t Q(t, s) du(s) + \int_t^b Q(t, s) du(s) \right| d\left(\bigvee_a^t(f)\right) \\
 &\leq \frac{1}{b-a} \int_a^b \left[\left| \int_a^t Q(t, s) du(s) \right| + \left| \int_t^b Q(t, s) du(s) \right| \right] d\left(\bigvee_a^t(f)\right) \\
 &= \frac{1}{b-a} \int_a^b [(b-t)|u(t) - u(a)| + (t-a)|u(b) - u(t)|] d\left(\bigvee_a^t(f)\right) \\
 &=: P.
 \end{aligned}$$

Now, on making use of the condition (4.1), we can state that

$$\begin{aligned}
 (4.4) \quad P &\leq \frac{1}{b-a} \int_a^b [L_a(b-t)(t-a)^\alpha + L_b(t-a)(b-t)^\beta] d\left(\bigvee_a^t(f)\right) \\
 &= \frac{1}{b-a} \left[L_a \int_a^b (b-t)(t-a)^\alpha d\left(\bigvee_a^t(f)\right) \right. \\
 &\quad \left. + L_b \int_a^b (t-a)(b-t)^\beta d\left(\bigvee_a^t(f)\right) \right].
 \end{aligned}$$

However,

$$\begin{aligned}
 &\int_a^b (b-t)(t-a)^\alpha d\left(\bigvee_a^t(f)\right) \\
 &= (b-t)(t-a)^\alpha \bigvee_a^t(f) \Big|_a^b - \int_a^b \left(\bigvee_a^t(f)\right) d[(b-t)(t-a)^\alpha] \\
 &= - \int_a^b \left(\bigvee_a^t(f)\right) [-(t-a)^\alpha + \alpha(b-t)(t-a)^{\alpha-1}] dt \\
 &= \int_a^b \left(\bigvee_a^t(f)\right) (t-a)^\alpha dt - \alpha \int_a^b \left(\bigvee_a^t(f)\right) (b-t)(t-a)^{\alpha-1} dt
 \end{aligned}$$

and

$$\begin{aligned}
& \int_a^b (t-a)(b-t)^\beta d\left(\bigvee_a^t(f)\right) \\
&= (t-a)(b-t)^\beta \bigvee_a^t(f) \Big|_a^b - \int_a^b \left(\bigvee_a^t(f)\right) d\left[(t-a)(b-t)^\beta\right] \\
&= - \int_a^b \left(\bigvee_a^t(f)\right) \left[(b-t)^\beta - \beta(t-a)(b-t)^{\beta-1}\right] dt \\
&= \beta \int_a^b \left(\bigvee_a^t(f)\right) (t-a)(b-t)^{\beta-1} dt - \int_a^b \left(\bigvee_a^t(f)\right) (b-t)^\beta dt
\end{aligned}$$

and from (4.4) we deduce the desired inequality (4.2). \square

Corollary 3. *If f is as in Theorem 4 and u is of $r-H$ -Hölder type, i.e.,*

$$(4.5) \quad |u(t) - u(s)| \leq H |u - t|^r \quad \text{for any } t, s \in [a, b],$$

where $H > 0$ and $r \in (0, 1)$ are given, then

$$\begin{aligned}
(4.6) \quad |D(f; u)| &\leq \frac{1}{b-a} H \int_a^b \left(\bigvee_a^t(f)\right) \left\{ (t-a)^r - (b-t)^r \right. \\
&\quad \left. + r(b-t)^{r-1}(t-a)^{r-1} \left[(t-a)^{1-r} - (b-t)^{1-r} \right] \right\} dt.
\end{aligned}$$

Remark 2. *If $r = \frac{1}{2}$ in Corollary 3, then we obtain the inequality:*

$$\begin{aligned}
(4.7) \quad |D(f; u)| &\leq \frac{1}{b-a} H \\
&\quad \times \int_a^b \left(\bigvee_a^t(f)\right) \left(\sqrt{t-a} - \sqrt{b-t}\right) \left(1 + \frac{1}{2\sqrt{(b-t)(t-a)}}\right) dt.
\end{aligned}$$

The following particular result may be useful for applications.

Corollary 4. *If f is as in Theorem 4 and $u : [a, b] \rightarrow \mathbb{R}$ is Lipschitzian with the constant $K > 0$, then*

$$\begin{aligned}
(4.8) \quad |D(f; u)| &\leq \frac{4}{b-a} \cdot K \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt \\
&\leq \begin{cases} K(b-a) \bigvee_a^b(f); \\ \frac{2(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} K \left(\int_a^b \left[\bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ 2K \int_a^b \left(\bigvee_a^t(f)\right) dt. \end{cases}
\end{aligned}$$

The multiplication constant 4 is best possible.

Proof. The first inequality follows by Theorem 4 on choosing $L_a = L_b = K$ and $\alpha = \beta = 1$.

Now, on utilising Hölder's inequality, we have

$$(4.9) \quad \int_a^b \left(t - \frac{a+b}{2} \right) \cdot \left(\bigvee_a^t(f) \right) dt \leq \begin{cases} \sup_{t \in [a, b]} \left(\bigvee_a^t(f) \right) \int_a^b \left| t - \frac{a+b}{2} \right| dt \\ \left(\int_a^b \left[\bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}} \left(\int_a^b \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \int_a^b \left(\bigvee_a^t(f) \right) dt \sup_{t \in [a, b]} \left| t - \frac{a+b}{2} \right|. \end{cases}$$

However, $\sup_{t \in [a, b]} \left| t - \frac{a+b}{2} \right| = \frac{b-a}{2}$ and

$$\begin{aligned} \int_a^b \left| t - \frac{a+b}{2} \right|^q dt &= 2 \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^q dt \\ &= \frac{(b-a)^{q+1}}{2^q (q+1)}, \quad q \geq 1 \end{aligned}$$

and by (4.9) we deduce:

$$(4.10) \quad \int_a^b \left(t - \frac{a+b}{2} \right) \cdot \bigvee_a^t(f) dt \leq \begin{cases} \frac{(b-a)^2}{4} \bigvee_a^t(f); \\ \frac{(b-a)^{1+\frac{1}{q}}}{2^{(q+1)\frac{1}{q}}} \left(\int_a^b \left[\bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} \int_a^b \left(\bigvee_a^t(f) \right) dt \end{cases}$$

and the second part is proved.

To prove the sharpness of the constant 4 in the first inequality in (4.8) assume that there exists $A > 0$ such that

$$(4.11) \quad |D(f; u)| \leq \frac{A}{b-a} \cdot K \int_a^b \left(t - \frac{a+b}{2} \right) \cdot \bigvee_a^t(f) dt,$$

provided that f is of bounded variation and u is K -Lipschitzian.

Let $f : [a, b] \rightarrow \mathbb{R}$,

$$f(t) = \begin{cases} 0 & \text{if } t \in [a, \frac{a+b}{2}], \\ k & \text{if } t \in (\frac{a+b}{2}, b], \end{cases}$$

with $k > 0$. Then

$$\bigvee_a^t(f) = \begin{cases} 0 & \text{if } t \in [a, \frac{a+b}{2}] , \\ k & \text{if } t \in (\frac{a+b}{2}, b] . \end{cases}$$

Also, we have

$$\begin{aligned} \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt &= \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) k dt \\ &= \frac{k(b-a)^2}{8} . \end{aligned}$$

Consider $u : [a, b] \rightarrow \mathbb{R}$, $u(t) = |t - \frac{a+b}{2}|$. Then u is K -Lipschitzian with $K = 1$. Also,

$$\begin{aligned} D(f; u) &= \int_a^b f(t) du(t) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t) dt \\ &= k \int_{\frac{a+b}{2}}^b du(t) = k \left[u(b) - u\left(\frac{a+b}{2}\right) \right] \\ &= \frac{(b-a)k}{2} . \end{aligned}$$

Substituting these values into (4.11) produces the inequality

$$\frac{(b-a)k}{2} \leq \frac{A}{b-a} \cdot \frac{k(b-a)^2}{8} ,$$

which implies that $A \geq 4$. □

5. INEQUALITIES FOR (l, L) -LIPSCHITZIAN FUNCTIONS

The following simple lemma holds.

Lemma 4. *Let $u : [a, b] \rightarrow \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

- (i) *The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a, b]$ is $\frac{1}{2}(L-l)$ -Lipschitzian;*
- (ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each } t, s \in [a, b], t \neq s;$$

- (iii) *We have the inequalities*

$$l(t-s) \leq u(t) - u(s) \leq L(t-s) \quad \text{for each } t, s \in [a, b] \text{ with } t > s.$$

The proof is obvious and we omit the details.

Definition 1 (see also [10]). *The function $u : [a, b] \rightarrow \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4 is said to be (l, L) -Lipschitzian on $[a, b]$. If $L > 0$ and $l = -L$, then $(-L, L)$ -Lipschitzian means L -Lipschitzian in the classical sense.*

The following result can be stated.

Theorem 5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation and $u : [a, b] \rightarrow \mathbb{R}$ an (l, L) -Lipschitzian function. Then*

$$(5.1) \quad |D(f; u)| \leq \frac{2}{b-a} (L-l) \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt$$

$$\leq \begin{cases} \frac{1}{2} (L-l) (b-a) \bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} (L-l) \left(\int_a^b \left[\bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (L-l) \int_a^b \left(\bigvee_a^t(f) \right) dt. \end{cases}$$

The constant 2 in the first inequality is sharp.

Proof. Observe that:

$$\begin{aligned} & D\left(f; u - \frac{l+L}{2} \cdot e\right) \\ &= \int_a^b \left(f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right) d\left[u(t) - \frac{l+L}{2} \cdot t \right] \\ &= \int_a^b \left(f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right) du(t) \\ &\quad - \frac{l+L}{2} \int_a^b \left(f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right) dt \\ &= D(f; u). \end{aligned}$$

Now, applying Corollary 4 for the function $u - \frac{l+L}{2}e$, which is $\frac{1}{2}(L-l)$ -Lipschitzian, we get:

$$\begin{aligned} \left| D\left(f; u - \frac{l+L}{2}e\right) \right| &\leq \frac{4}{b-a} \cdot \frac{1}{2} (L-l) \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt \\ &= \frac{2}{b-a} (L-l) \int_a^b \left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t(f) dt \end{aligned}$$

and the theorem is proved. \square

The second result may be stated as:

Theorem 6. Let $u : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation. If $f : [a, b] \rightarrow \mathbb{R}$ is (ϕ, Φ) -Lipschitzian with $\Phi > \phi$, then

$$\begin{aligned}
 (5.2) \quad & \left| D(f; u) - \frac{\phi + \Phi}{2} \left[\frac{u(b) + u(a)}{2} (b - a) - \int_a^b u(t) dt \right] \right| \\
 & \leq \frac{1}{2} (\Phi - \phi) \cdot \left[\frac{1}{2} (b - a) \bigvee_a^b(u) - \frac{2}{b - a} \int_a^b \left(\bigvee_a^s(u) \right) \left(s - \frac{a + b}{2} \right) ds \right] \\
 & \leq \frac{1}{4} \cdot (\Phi - \phi) (b - a) \bigvee_a^b(u).
 \end{aligned}$$

The constant $\frac{1}{2}$ in front of $(\Phi - \phi)$ and $\frac{1}{4}$ are best possible.

Proof. Observe that

$$\begin{aligned}
 & D\left(f - \frac{\phi + \Phi}{2} \cdot e; u\right) \\
 &= \int_a^b \left[f(t) - \frac{\phi + \Phi}{2} \cdot t - \frac{1}{b - a} \int_a^b \left(f(s) - \frac{\phi + \Phi}{2} \cdot s \right) ds \right] du(t) \\
 &= \int_a^b \left[f(t) - \frac{1}{b - a} \int_a^b f(s) ds - \left(\frac{\phi + \Phi}{2} t - \frac{1}{b - a} \int_a^b \frac{\phi + \Phi}{2} \cdot s ds \right) \right] du(t) \\
 &= D(f; u) - \frac{\phi + \Phi}{2} \int_a^b \left(t - \frac{1}{b - a} \int_a^b s ds \right) du(t) \\
 &= D(f; u) - \frac{\phi + \Phi}{2} \int_a^b \left(t - \frac{a + b}{2} \right) du(t).
 \end{aligned}$$

Since, integrating by parts in the Riemann-Stieltjes integral we have:

$$\int_a^b \left(t - \frac{a + b}{2} \right) du(t) = \frac{u(b) + u(a)}{2} (b - a) - \int_a^b u(t) dt,$$

then

$$D\left(f - \frac{\phi + \Phi}{2} e; u\right) = D(f; u) - \frac{\phi + \Phi}{2} \left[\frac{u(b) + u(a)}{2} (b - a) - \int_a^b u(t) dt \right].$$

Now, on applying Theorem 2 for the function $f - \frac{\phi + \Phi}{2} e$ which is $\frac{1}{2} (L - l)$ -Lipschitzian, we deduce the desired result (5.2). \square

6. APPLICATIONS FOR THE ČEBYŠEV FUNCTIONAL

If we choose $u(t) := \int_a^t g(\tau) d\tau$, $t \in [a, b]$, where $g : [a, b] \rightarrow \mathbb{R}$ is Lebesgue integrable on $[a, b]$, then we have the equality

$$C(f; g) = \frac{1}{b - a} D(f; u).$$

Also, u is of bounded variation on any subinterval $[a, s]$, $s \in [a, b]$ and if g is continuous on $[a, b]$, then

$$\bigvee_a^s(u) = \int_a^s |g(\tau)| d\tau, \quad s \in [a, b].$$

If f is of bounded variation on $[a, b]$, then on utilising the inequality (3.5) we have

$$\begin{aligned}
 (6.1) \quad |C(f; g)| &\leq \frac{1}{(b-a)^2} \left[\int_a^b (2s-a-b) |g(s)| \bigvee_a^s(f) ds \right. \\
 &\quad \left. + 2 \int_a^b \left(\int_a^s |g(\tau)| d\tau \right) \bigvee_a^s(f) ds - \int_a^b |g(\tau)| d\tau \cdot \int_a^b \left(\bigvee_a^s(f) \right) ds \right] \\
 &\leq \frac{1}{(b-a)^2} \int_a^b (2s-a-b) |g(s)| \bigvee_a^s(f) ds \\
 &\quad + \frac{1}{(b-a)^2} \int_a^b \left(\int_a^s |g(\tau)| d\tau \right) \bigvee_a^s(f) ds \\
 &\leq \frac{1}{(b-a)^2} \int_a^b (2s-a-b) |g(s)| \bigvee_a^s(f) ds + \frac{1}{b-a} \int_a^b |g(\tau)| d\tau \cdot \bigvee_a^b(f).
 \end{aligned}$$

Now, if f is monotonic nondecreasing, then by (3.9) we have

$$\begin{aligned}
 (6.2) \quad |C(f; g)| &\leq \frac{1}{(b-a)^2} \left[\int_a^b (2s-a-b) f(s) |g(s)| ds \right. \\
 &\quad \left. + 2 \int_a^b \left(\int_a^s |g(\tau)| d\tau \right) f(s) ds - \int_a^b f(s) ds \cdot \int_a^b |g(\tau)| d\tau \right].
 \end{aligned}$$

The case where f is L -Lipschitzian provides via (3.8) a much simpler inequality:

$$\begin{aligned}
 (6.3) \quad |C(f; g)| &\leq L \left[\frac{1}{2} \int_a^b |g(\tau)| d\tau - \frac{2}{(b-a)^2} \int_a^b \left(\int_a^s |g(\tau)| d\tau \right) \left(s - \frac{a+b}{2} \right) ds \right] \\
 &\leq \frac{1}{2} L \int_a^b |g(s)| ds.
 \end{aligned}$$

Now, if f is of bounded variation and $|g|$ is bounded above by M , i.e., $|g(t)| \leq M$ for a.e. $t \in [a, b]$, then by (4.8) we have:

$$\begin{aligned}
 (6.4) \quad |C(f; g)| &\leq \frac{4}{(b-a)^2} M \int_a^b \left(t - \frac{a+b}{2} \right) \bigvee_a^t(f) dt \\
 &\leq \begin{cases} M \bigvee_a^b(f); \\ \frac{2(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} M \left(\int_a^b \left[\bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{2M}{b-a} \int_a^b \left(\bigvee_a^t(f) \right) dt. \end{cases}
 \end{aligned}$$

The constant 4 in (6.4) is best possible.

Finally, if $-\infty < \phi \leq g(t) \leq \Phi$ for a.e. $t \in [a, b]$, then $\left|g(t) - \frac{\phi + \Phi}{2}\right| \leq \frac{1}{2}(\Phi - \phi)$ and since

$$C\left(f; g - \frac{\phi + \Phi}{2}\right) = C(f; g),$$

hence, by (6.4) we deduce the inequalities

$$(6.5) \quad |C(f; g)| \leq \frac{2}{(b-a)^2} (\Phi - \phi) \int_a^b \left(t - \frac{a+b}{2}\right) \bigvee_a^t(f) dt$$

$$\leq \begin{cases} \frac{1}{2} (\Phi - \phi) \bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} (\Phi - \phi) \left(\int_a^b \left[\bigvee_a^t(f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\Phi - \phi}{b-a} \int_a^b \left(\bigvee_a^t(f) \right) dt. \end{cases}$$

The constant 2 in the first inequality is best possible.

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